Fibonacci Heaps

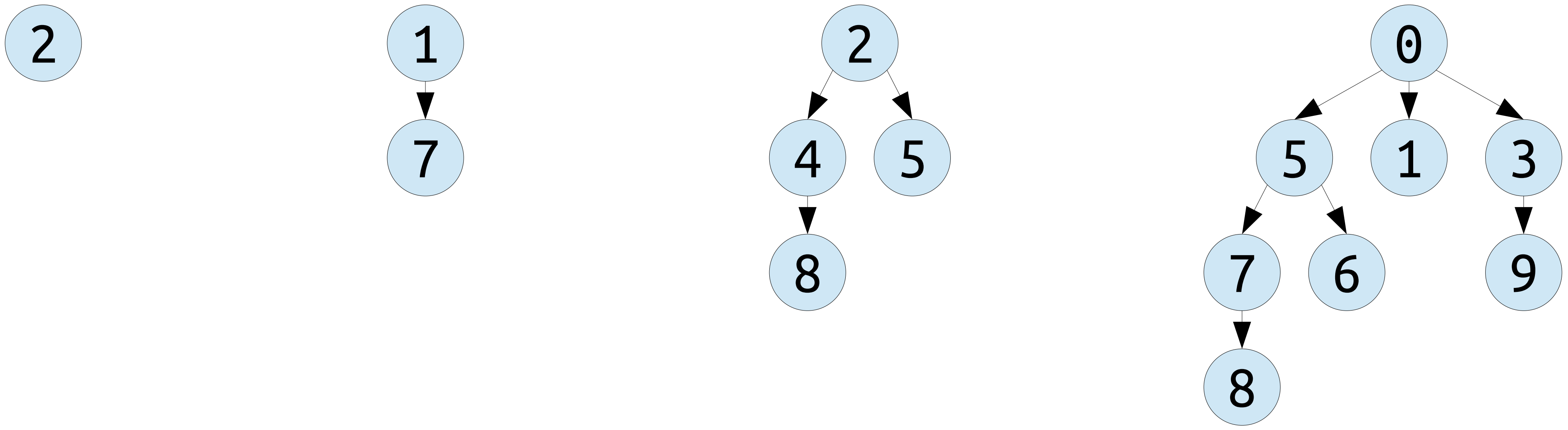
Outline for Today

- ***Recap from Last Time***
  - Quick refresher on binomial heaps and lazy binomial heaps.
- ***The Need for decrease-key***
  - An important operation in many graph algorithms.
- ***Fibonacci Heaps***
  - A data structure efficiently supporting ***decrease-key***.
- ***Representational Issues***
  - Some of the challenges in Fibonacci heaps.

Recap from Last Time

(Lazy) Binomial Heaps

- Last time, we covered the ***binomial heap*** and a variant called the ***lazy binomial heap***.
- These are priority queue structures designed to support efficient ***meld***ing.
- Elements are stored in a collection of ***binomial trees***.



Operation Costs

- Each individual ***extract-min*** in a lazy binomial heap may take a while, but amortizes out to O(log n).
- ***Intuition:*** Each ***extract-min*** does cleanup for the earlier ***enqueue*** operations, leaving the heap with few trees.

Eager Binomial Heap:

- ***enqueue***: O(log n)
- ***meld***: O(log n)
- ***find-min***: O(log n)
- ***extract-min***: O(log n)

Lazy Binomial Heap:

- ***enqueue***: O(1)
- ***meld***: O(1)
- ***find-min***: O(1)
- ***extract-min***: O(log n)*

**amortized*

New Stuff!

The Need for ***decrease-key***

The ***decrease-key*** Operation

- Some priority queues support the operation ***decrease-key***(v, k), which works as follows:

Given a pointer to an element v, lower its key (priority) to k. It is assumed that k is less than the current priority of v.

- This operation is crucial in efficient implementations of Dijkstra's algorithm and Prim's MST algorithm.

Dijkstra and ***decrease-key***

- Dijkstra's algorithm can be implemented with a priority queue using
  - O(n) total ***enqueue***s,
  - O(n) total ***extract-min***s, and
  - O(m) total ***decrease-key***s.
- Dijkstra's algorithm runtime is

$$\mathbf{O}(\boldsymbol{n}\ \mathbf{T}_{\mathbf{enq}} + \boldsymbol{n}\ \mathbf{T}_{\mathbf{ext}} + \boldsymbol{m}\ \mathbf{T}_{\mathbf{dec}})$$

Prim and ***decrease-key***

- Prim's algorithm can be implemented with a priority queue using
  - O(n) total ***enqueue***s,
  - O(n) total ***extract-min***s, and
  - O(m) total ***decrease-key***s.
- Prim's algorithm runtime is

$$\mathbf{O(\boldsymbol{n}\ T_{enq} + \boldsymbol{n}\ T_{ext} + \boldsymbol{m}\ T_{dec})}$$

Standard Approaches

- In a binary heap, ***enqueue***, ***extract-min***, and ***decrease-key*** can be made to work in time O(log n) time each.
- Cost of Dijkstra's / Prim's algorithm:

$$O(n\,T_{\text{enq}} + n\,T_{\text{ext}} + m\,T_{\text{dec}})$$

$$= O(n \log n + n \log n + m \log n)$$

$$= \mathbf{O(\boldsymbol{m} \log \boldsymbol{n})}$$

Standard Approaches

- In a lazy binomial heap, ***enqueue*** takes amortized time O(1), and ***extract-min*** and ***decrease-key*** take amortized time O(log n).
- Cost of Dijkstra's / Prim's algorithm:

$$O(n\,T_{\text{enq}} + n\,T_{\text{ext}} + m\,T_{\text{dec}})$$

$$= O(n + n \log n + m \log n)$$

$$= \mathbf{O(\boldsymbol{m} \log \boldsymbol{n})}$$

Where We're Going

- The ***Fibonacci heap*** has these amortized runtimes:
  - ***enqueue***: O(1)
  - ***extract-min***: O(log *n*).
  - ***decrease-key***: O(1).
- Cost of Prim's or Dijkstra's algorithm:

$$O(n\ T_{\text{enq}} + n\ T_{\text{ext}} + m\ T_{\text{dec}})$$

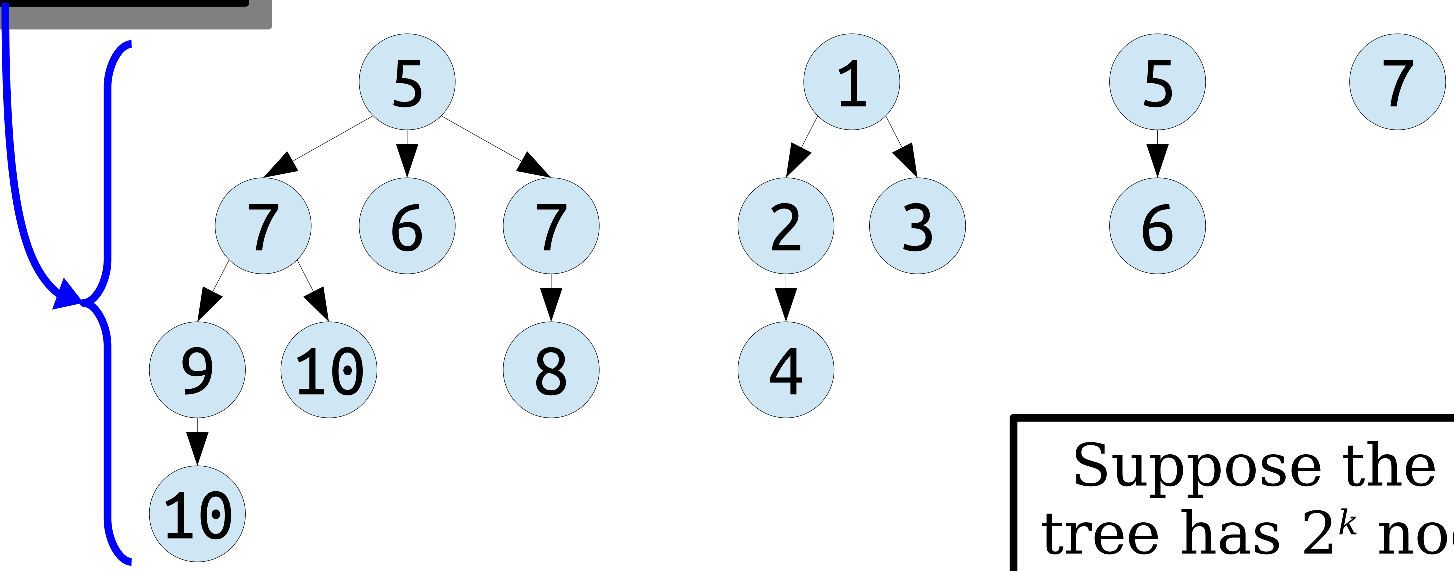
$$= O(n + n \log n + m)$$

$$= \mathbf{O(m + n \log n)}$$

- This is theoretically optimal for a comparison-based priority queue in Dijkstra's or Prim's algorithms.

The Challenge of ***decrease-key***

If our lazy binomial heap has n nodes, how tall can the tallest tree be?

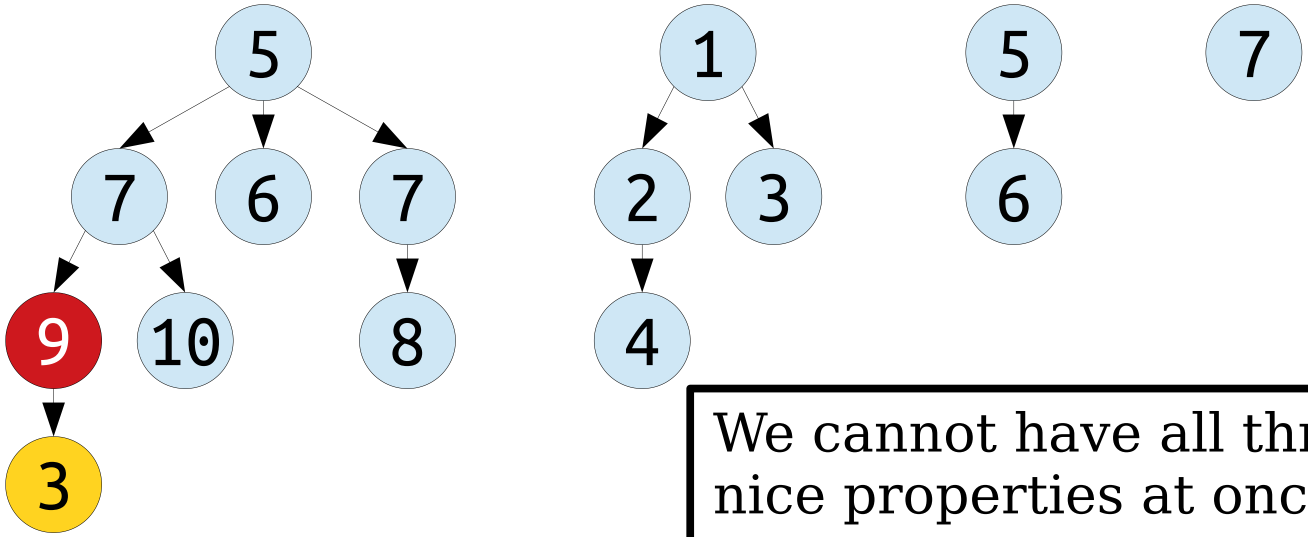


Suppose the biggest tree has 2^k nodes in it.

Then $2^k \leq n$.

So $k = O(\log n)$.

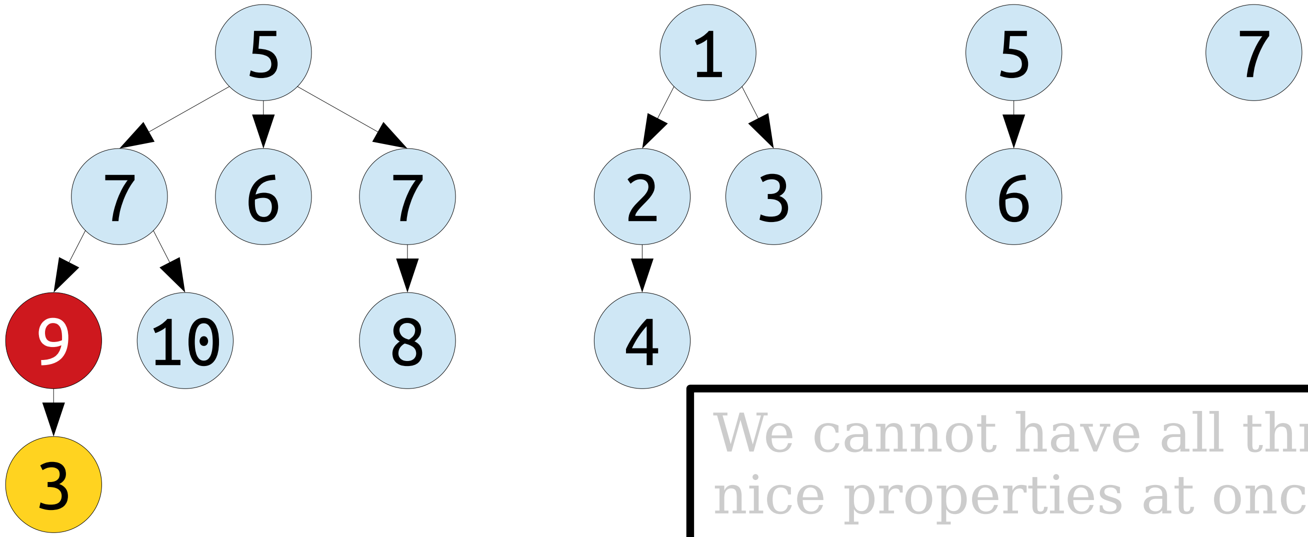
Challenge: Support ***decrease-key*** in (amortized) time O(1).



We cannot have all three of these nice properties at once:

1. ***decrease-key*** takes time O(1).
2. Our trees are heap-ordered.
3. Our trees are binomial trees.

Challenge: Support ***decrease-key*** in (amortized) time O(1).



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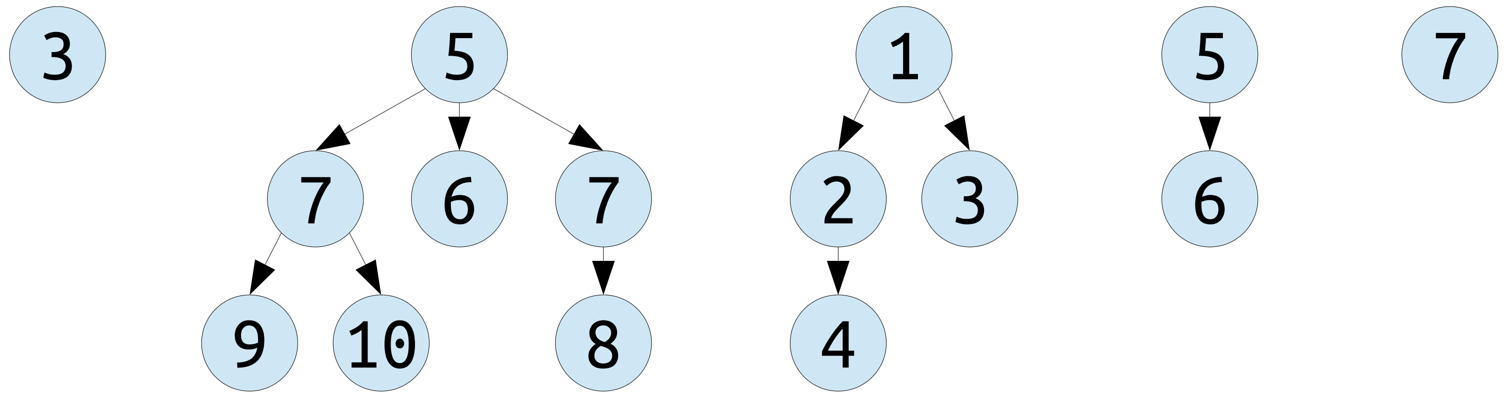

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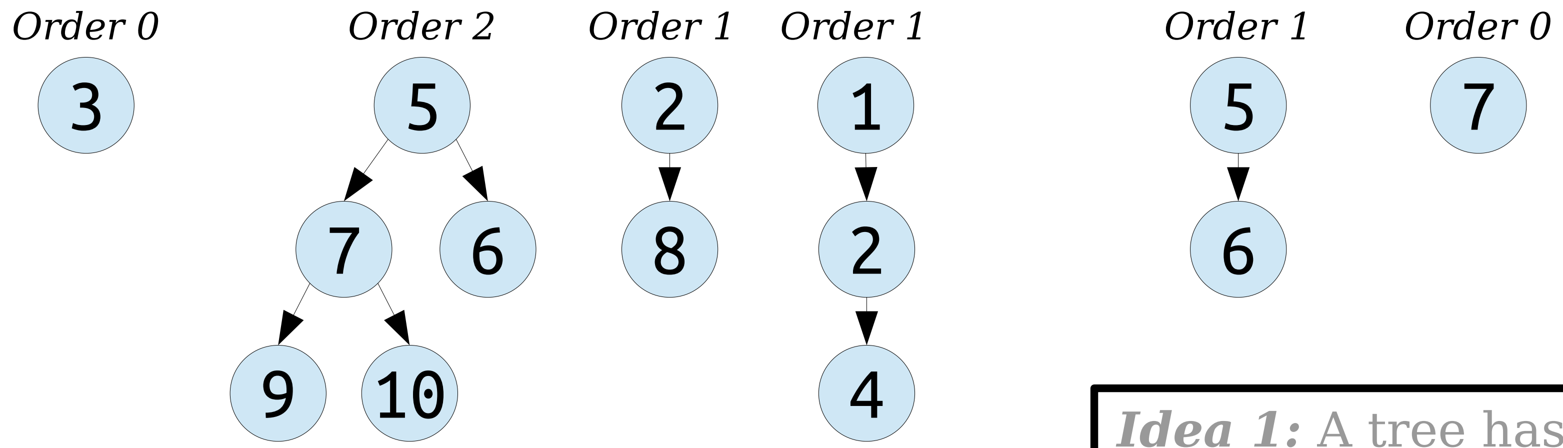
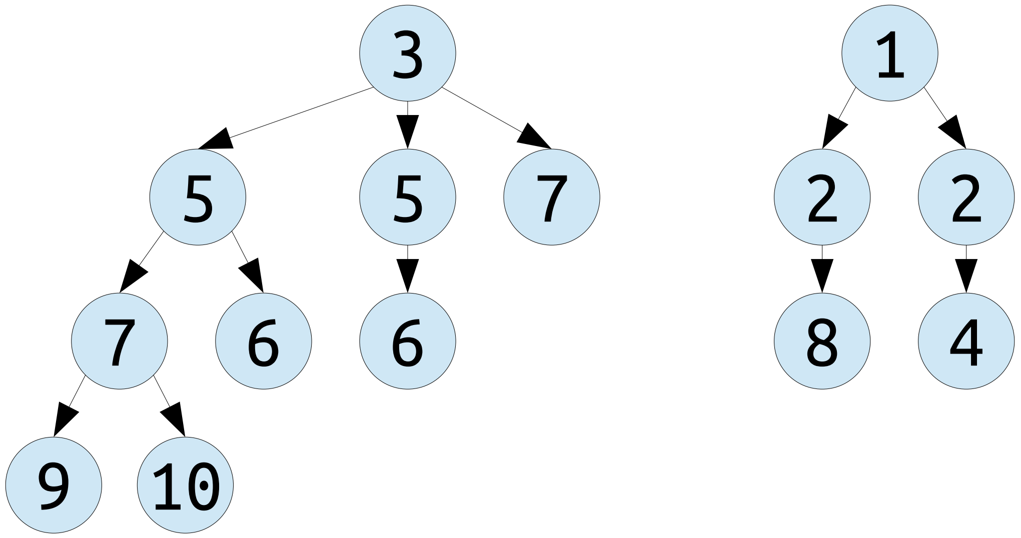


Idea 1: A tree has order k if it has 2^k nodes.

Idea 2: A tree has order k if its root has k children.

Problem: What do we do in an ***extract-min***?

Question: How efficient is this?

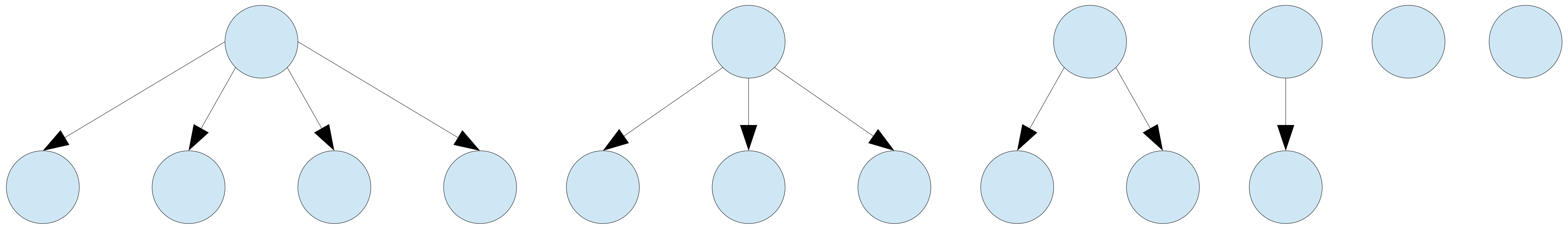


(1) To do a ***decrease-key***, cut the node from its parent.
(2) Do ***extract-min*** as usual, using child count as order.

Intuition: ***extract-min*** is only fast if it compacts nodes into a few trees.

There are $\Theta(n^{1/2})$ trees here.

What happens if we repeatedly ***enqueue*** and ***extract-min*** a small value?



Claim: Because tree shapes aren't well-constrained, we can force ***extract-min*** to take amortized time $\Omega(n^{1/2})$.

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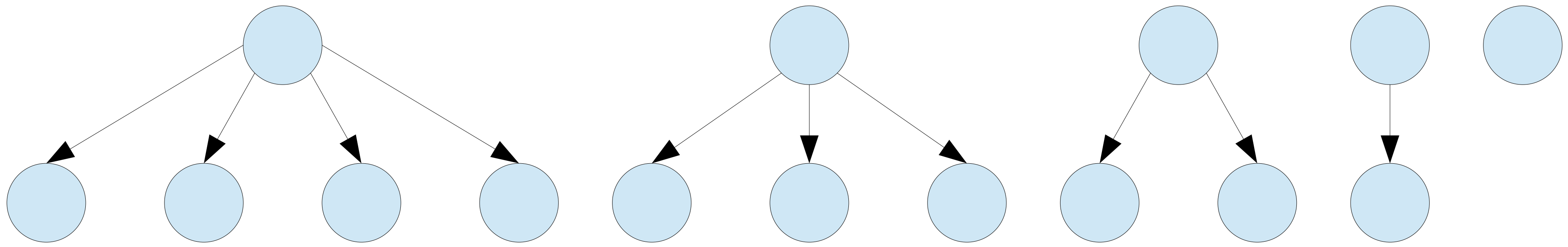
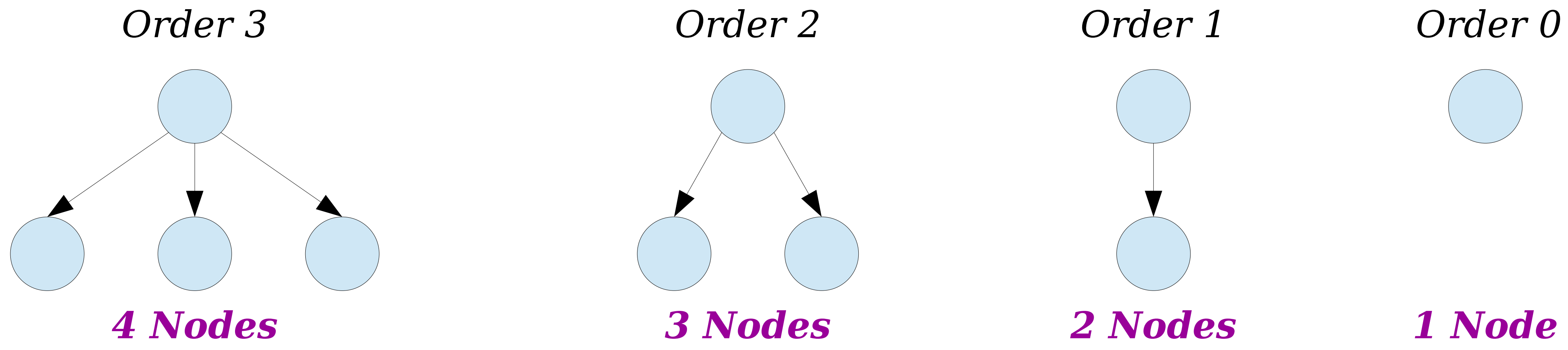


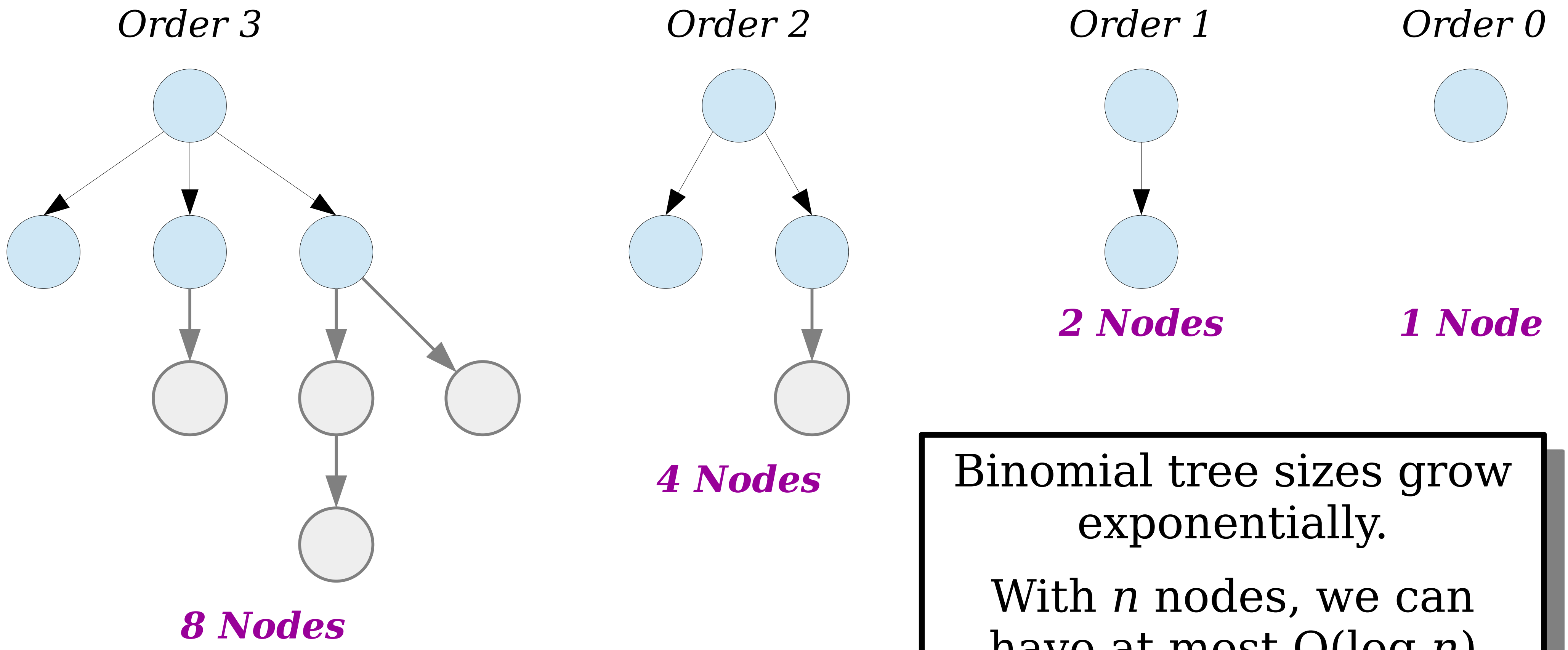
Each operation does $\Theta(n^{1/2})$ work, and doesn't make any future operations any better.

Claim: Because tree shapes aren't well-constrained, we can force ***extract-min*** to take amortized time $\Omega(n^{1/2})$.



With n nodes, it's possible to have $\Omega(n^{1/2})$ trees of distinct orders.

Question: Why didn't this happen before?



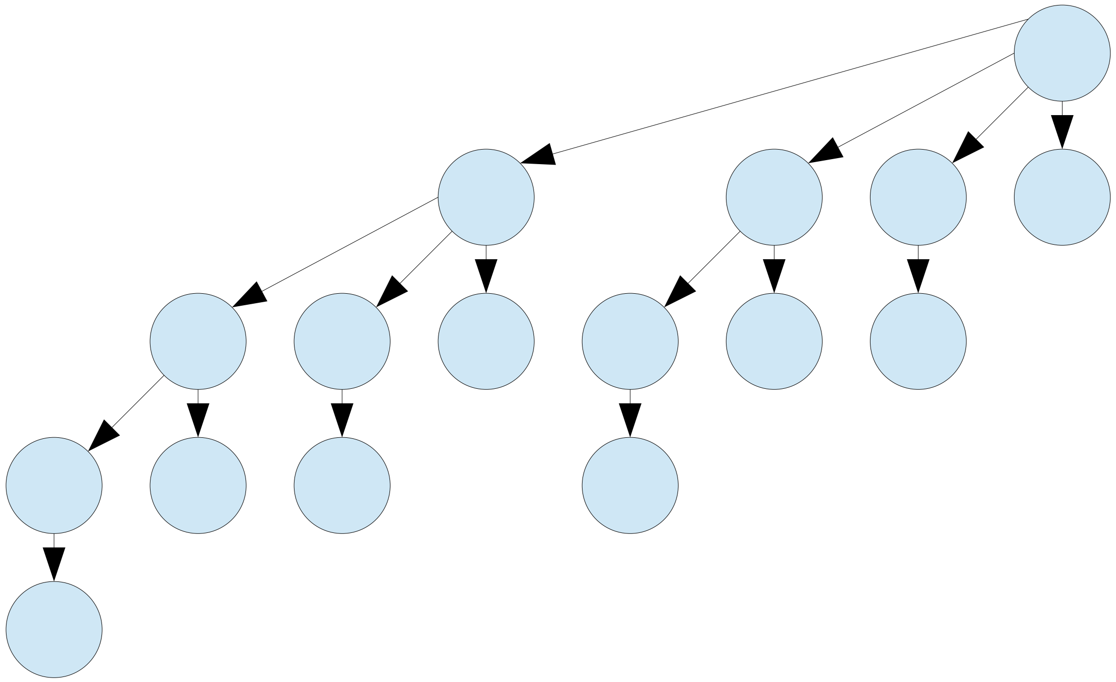
Binomial tree sizes grow exponentially.

With n nodes, we can have at most O(log n) trees of distinct orders.

Question: Why didn't this happen before?

Intuition: Allow trees to get somewhat imbalanced, slowly propagating information to the root.

Rule: Nodes can lose at most one child. If a node loses two children, cut it from its parent.



Goal: Make tree sizes grow exponentially with order, but still allow for subtrees to be cut out quickly.

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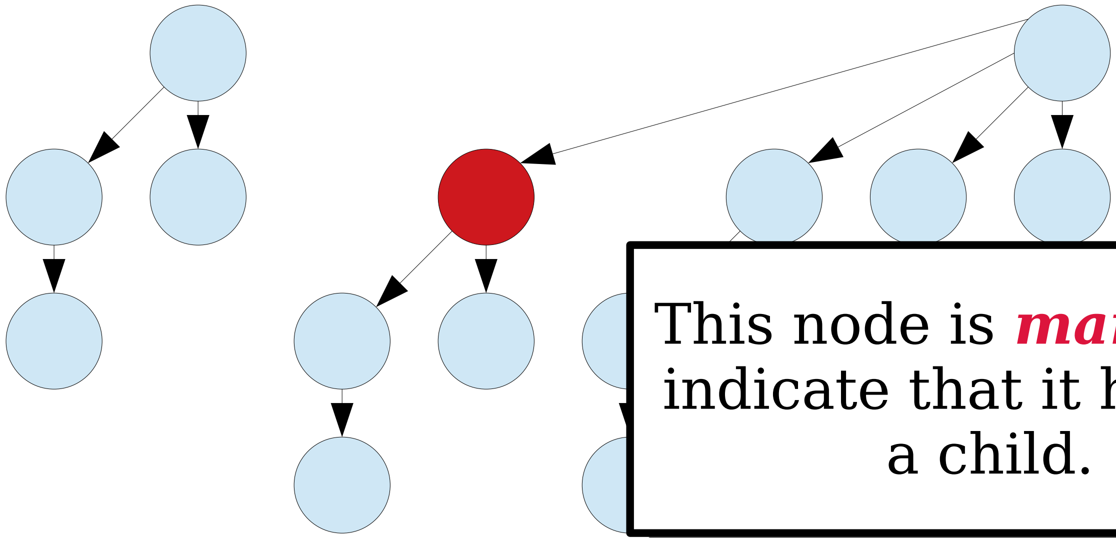
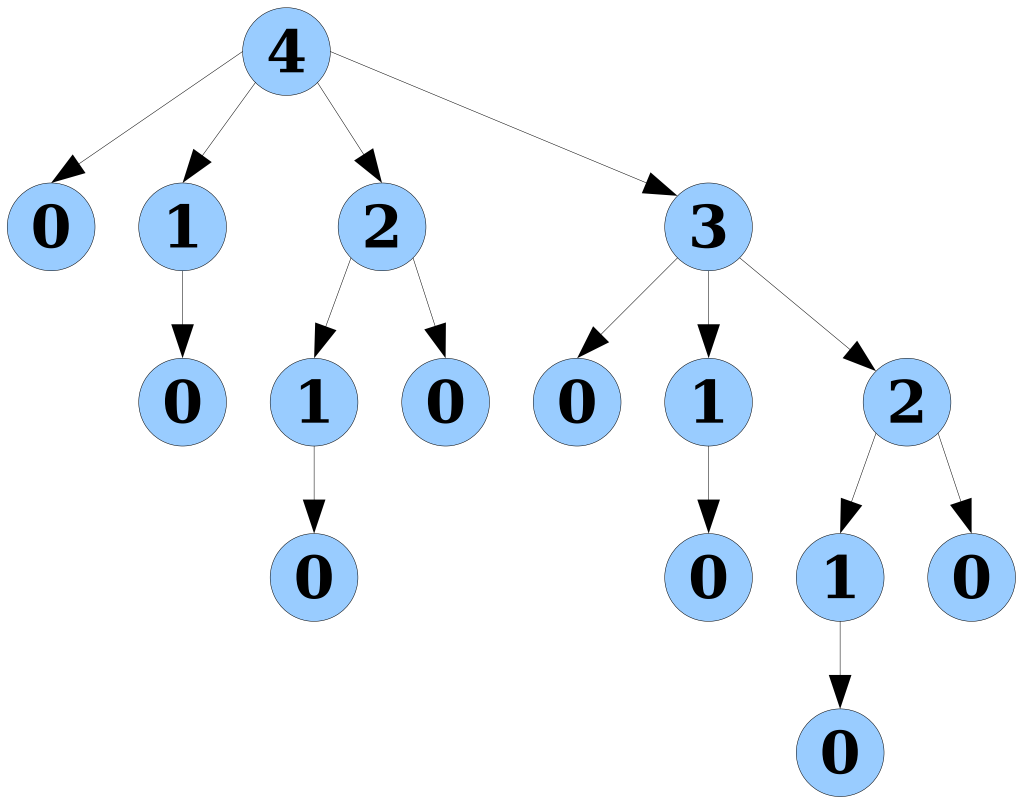


This node is ***marked*** to indicate that it has lost a child.

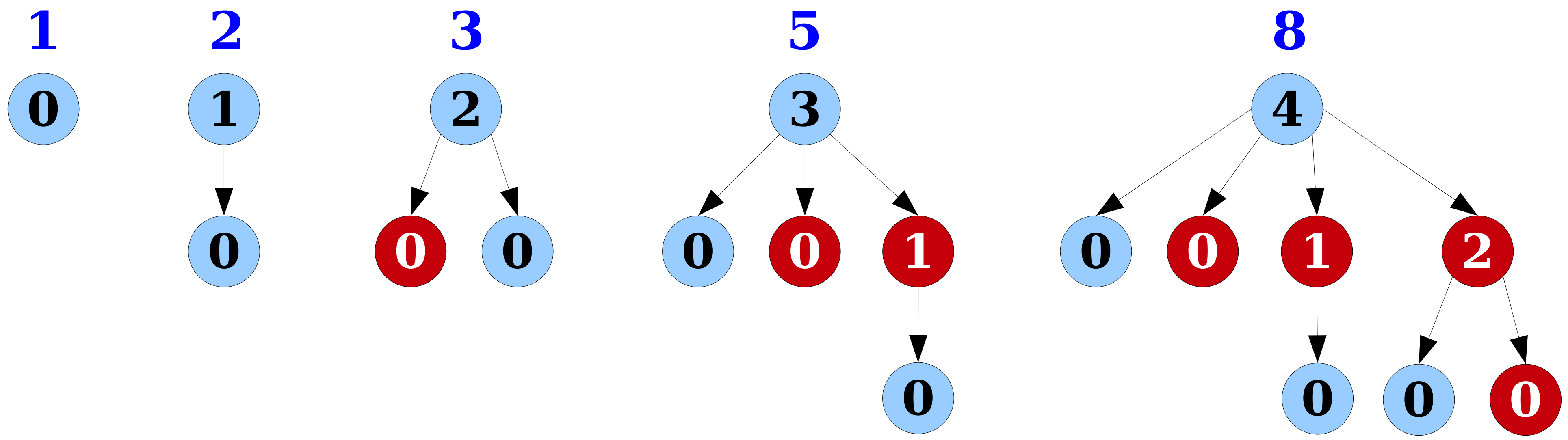
Goal: Make tree sizes grow exponentially with order, but still allow for subtrees to be cut out quickly.

Maximally-Damaged Trees

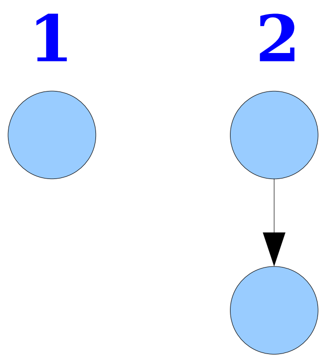
- Here's a binomial tree of order 4. That is, the root has four children.
- ***Question:*** Using our marking scheme, how many nodes can we remove without changing the order of the tree?
- Equivalently: how many nodes can we remove without removing any direct children of the root?



Maximally-Damaged Trees



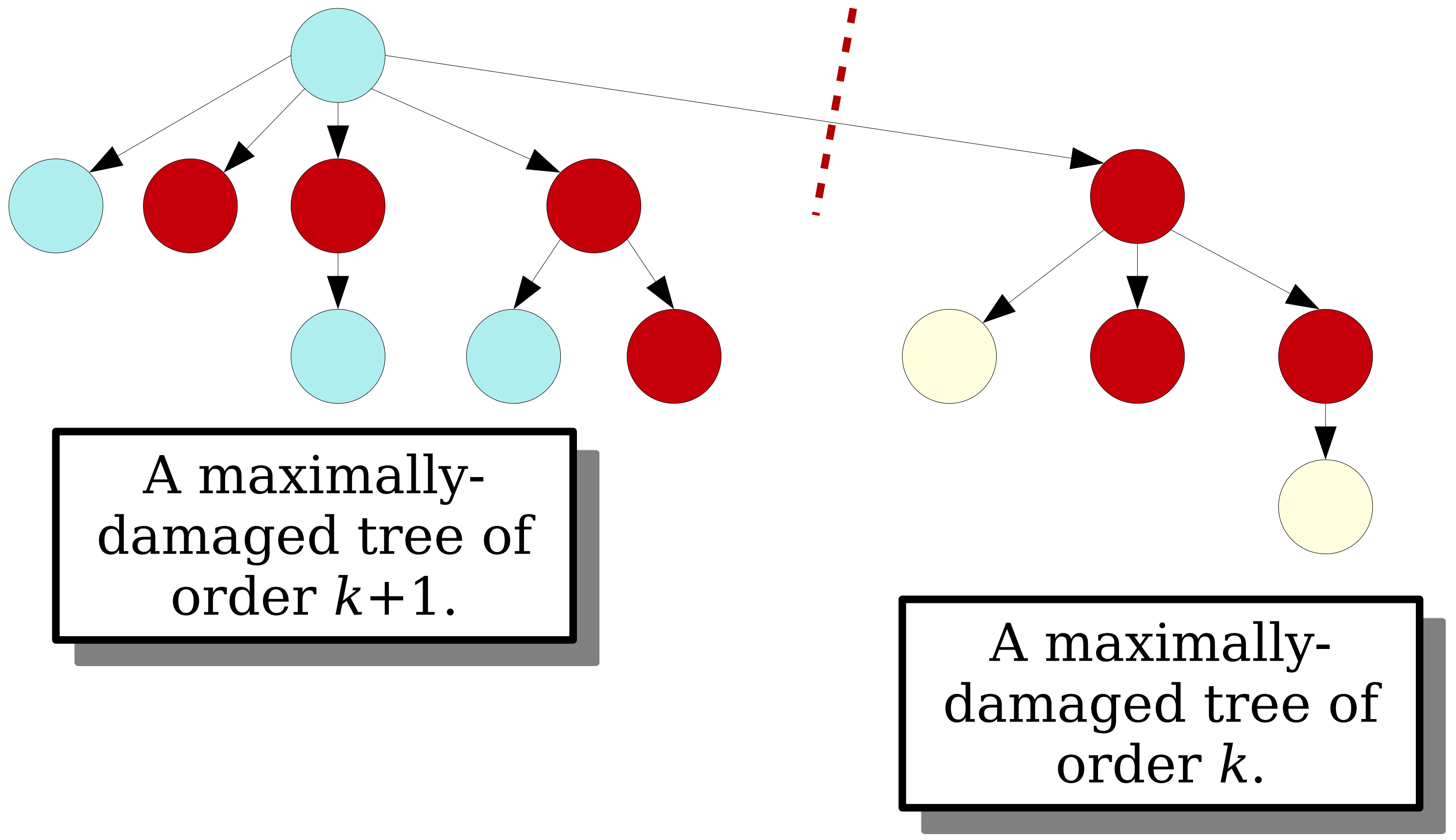
Claim: The minimum number of nodes in a tree of order k is $\boldsymbol{F_{k+2}}$



These trees are the base cases for our inductive line of reasoning.

Theorem: The minimum number of nodes in a tree of order k is $\boldsymbol{F_{k+2}}$.

Thanks to former CS166ers Kevin Tan and Max Arseneault for this proof approach!


A maximally-damaged tree of order k+1.
A maximally-damaged tree of order k.

Theorem: The minimum number of nodes in a tree of order k is $\boldsymbol{F_{k+2}}$.

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Fact: $F_k = \Theta(\varphi^k)$, where

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is the golden ratio.

Corollary: The number of nodes in a tree of order k grows exponentially with k (approximately 1.61^k versus our previous 2^k).

Theorem: The minimum number of nodes in a tree of order k is $\boldsymbol{F_{k+2}}$.

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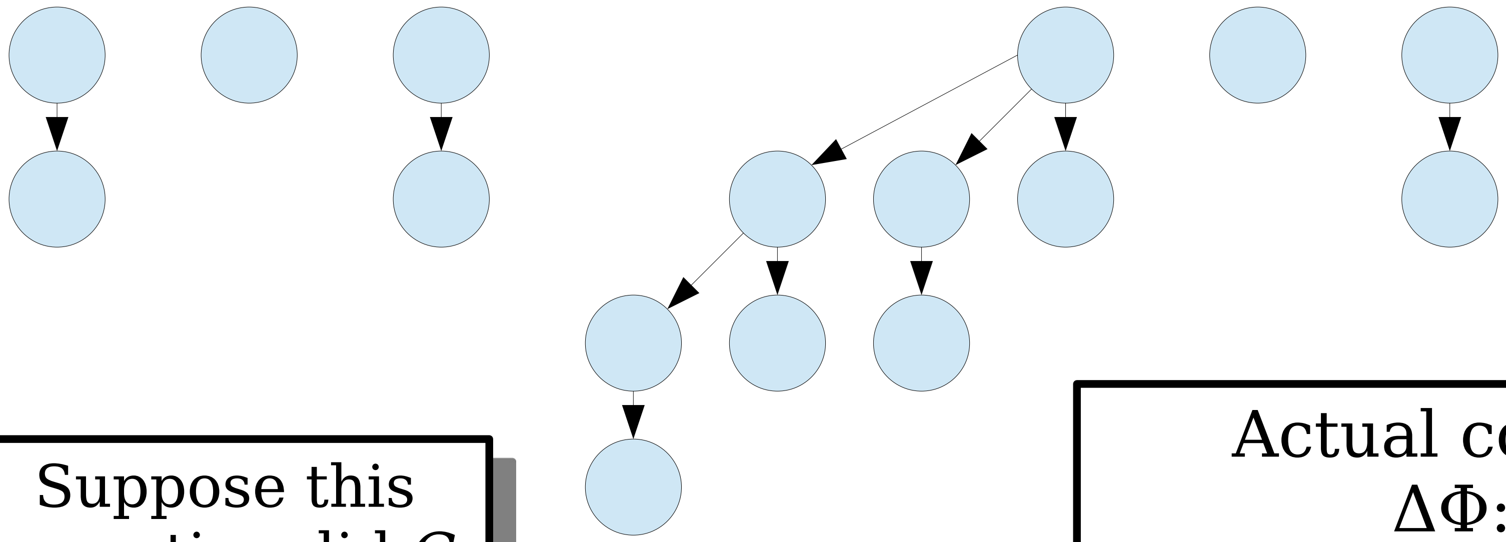
A ***Fibonacci heap*** is a lazy binomial heap with ***decrease-key*** implemented using the “lose at most one child” marking scheme.

How fast are the operations on Fibonacci heaps?

$$\Phi = t + m$$

where

t is the number of trees and
m is the number of marked nodes.



Suppose this operation did C total cuts.

Actual cost: O(C)
$\Delta\Phi$: +1

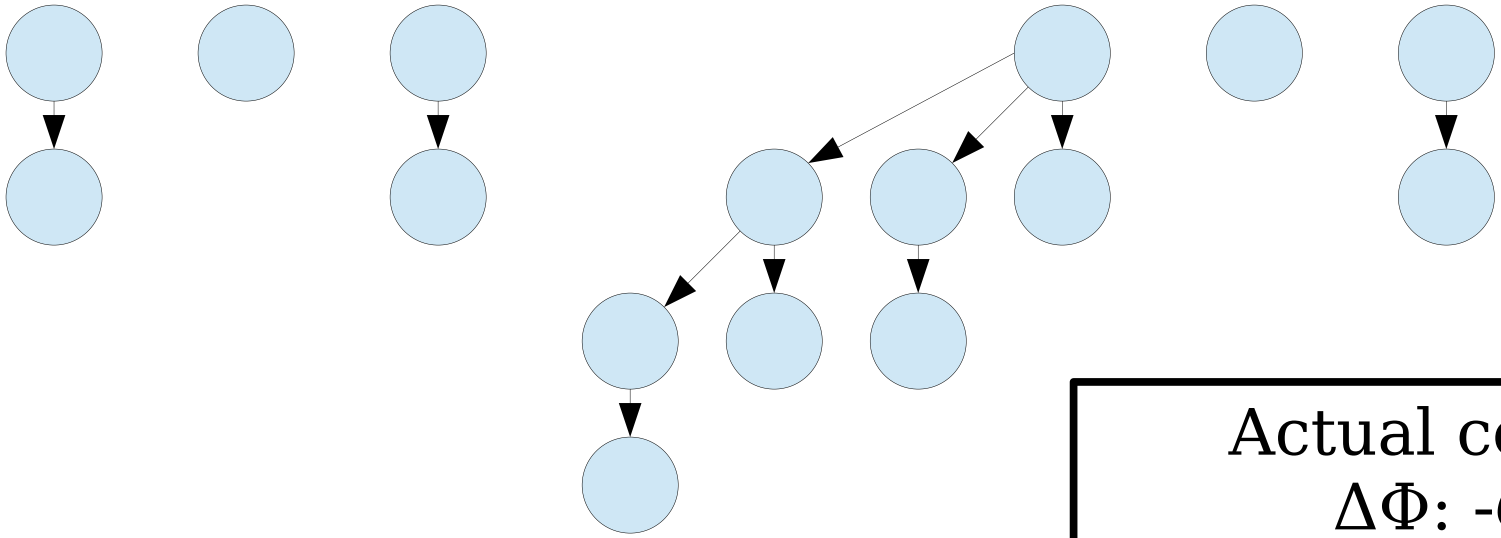
Amortized cost: **O(C)**.

Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

$$\Phi = t + 2m$$

where

t is the number of trees and
m is the number of marked nodes.



Actual cost: O(C)
ΔΦ: -C + 1

Amortized cost: **O(1)**.

Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

The Overall Analysis

- Here’s the final scorecard for the Fibonacci heap.
- These are excellent theoretical runtimes. There’s minimal room for improvement!
- Later work made all these operations *worst-case efficient* at a significant increase in both runtime and intellectual complexity.

enqueue: O(1)

find-min: O(1)

meld: O(1)

extract-min: O(log n)*

decrease-key: O(1)*

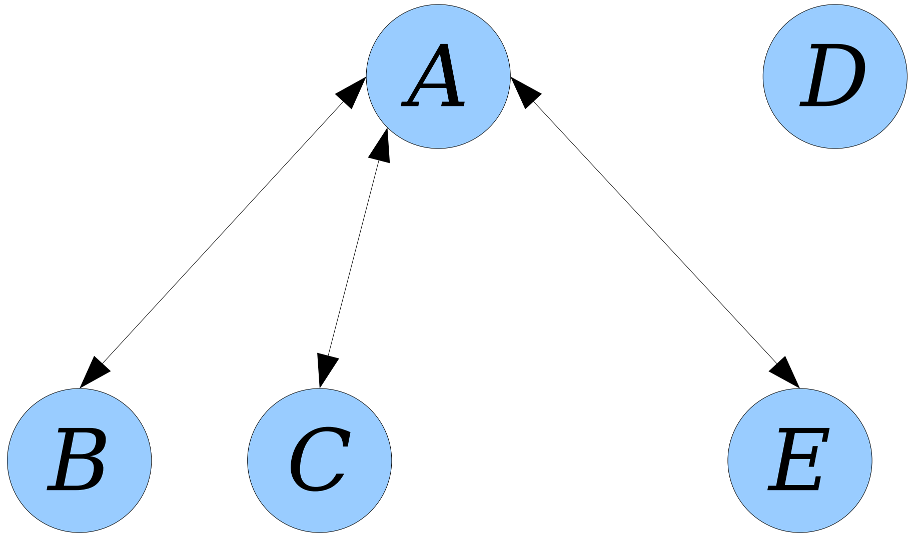
**amortized*

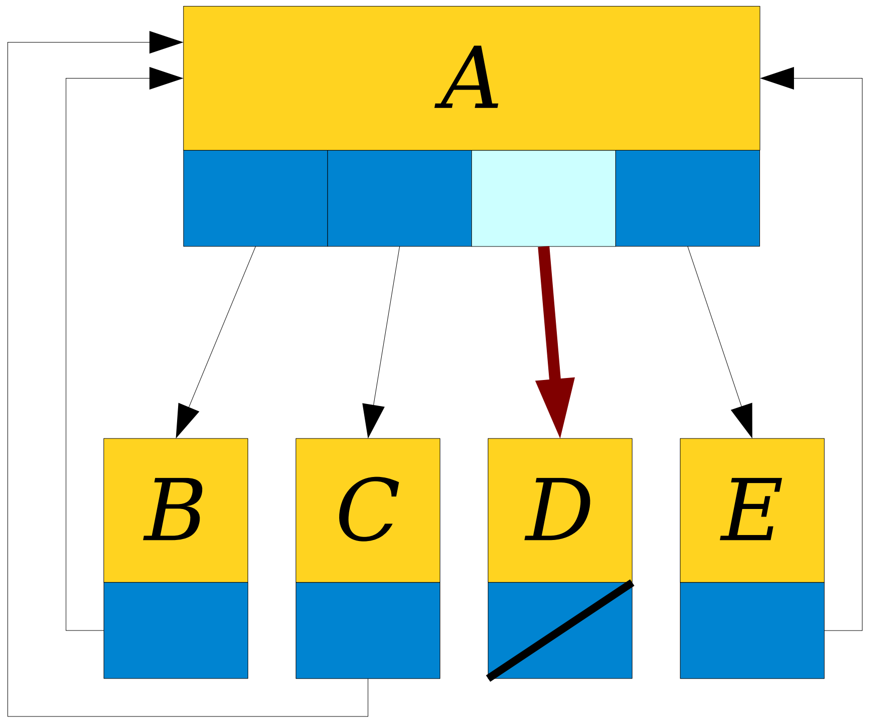
Representation Issues

Representing Trees

- The trees in a Fibonacci heap must be able to do the following:
  - During a merge: Add one tree as a child of the root of another tree.
  - During a cut: Cut a node from its parent in time O(1).
- ***Claim:*** This is trickier than it looks.

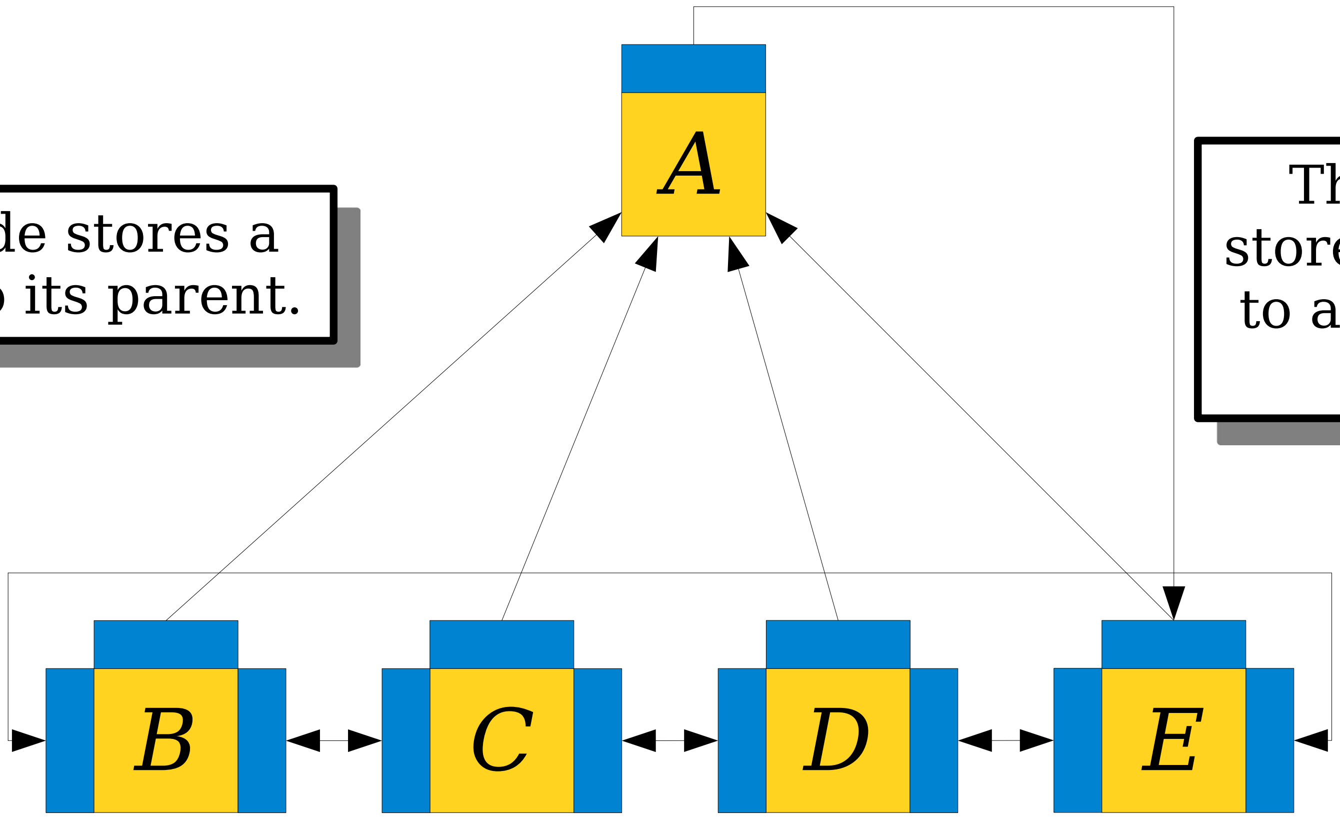
Representing Trees





Finding this pointer might take time Θ(log *n*)!

The Solution



Each node stores a pointer to its parent.

The parent stores a pointer to an arbitrary child.

The children of each node are in a circularly, doubly-linked list.

Awful Linked Lists

- Trees are stored as follows:
    - Each node stores a pointer to *some* child.
    - Each node stores a pointer to its parent.
    - Each node is in a circularly-linked list of its siblings.
- The following possible are now possible in time O(1):
    - Cut a node from its parent.
    - Add another child node to a node.

Fibonacci Heap Nodes

- Each node in a Fibonacci heap stores
  - A pointer to its parent.
  - A pointer to the next sibling.
  - A pointer to the previous sibling.
  - A pointer to an arbitrary child.
  - A bit for whether it's marked.
  - Its order.
  - Its key.
  - Its element.

In Practice

- In practice, the constant factors on Fibonacci heaps make it slower than other heaps, except on huge graphs or workflows with tons of ***decrease-key***s.
- Why?
  - Huge memory requirements per node.
  - High constant factors on all operations.
  - Poor locality of reference and caching.

In Theory

- That said, Fibonacci heaps are worth knowing about for several reasons:
  - Clever use of a two-tiered potential function shows up in lots of data structures.
  - Implementation of ***decrease-key*** forms the basis for many other advanced priority queues.
  - Gives the theoretically optimal comparison-based implementation of Prim's and Dijkstra's algorithms.

More to Explore

- Since the development of Fibonacci heaps, there have been a number of other priority queues with similar runtimes.
  - In 1986, a powerhouse team (Fredman, Sedgewick, Sleator, and Tarjan) invented the ***pairing heap***. It's much simpler than a Fibonacci heap, is fast in practice, but its runtime bounds are unknown!
  - In 2012, Brodal et al. invented the ***strict Fibonacci heap***. It has the same time bounds as a Fibonacci heap, but in a *worst-case* rather than *amortized* sense.
  - In 2013, Chan invented the ***quake heap***. It matches the asymptotic bounds of a Fibonacci heap but uses a totally different strategy.
- Also interesting to explore: if the weights on the edges in a graph are chosen from a continuous distribution, the expected number of ***decrease-key***s in Dijkstra's algorithm is $O(n \log (m / n))$. That might counsel another heap structure!
- Also interesting to explore: binary heaps generalize to b-ary heaps, where each node has b children. Picking $b = \log (2 + {}^{m}/_{n})$ makes Dijkstra and Prim run in time $O(m \log n / \log {}^{m}/_{n})$, which is $O(m)$ if $m = \Theta(n^{1+\varepsilon})$ for any $\varepsilon > 0$.
- Recent result: Dijkstra's algorithm with Fibonacci heaps can be combined with other data structures to be ***instance optimal*** for single-source shortest paths; *any* algorithm for solving SSSP is at most a constant factor faster than Dijkstra's plus the modified heap.

Upcoming Topics!

Hypergraph Cores and Peeling

Major improvements to hash tables, databases, and filters.

Cute Tricks with Integers

Harnessing latent parallelism that was there all along.

Data Structures in 2+ Dimensions

Searching in polygons, hypercubes, etc.

The Quest for the Best BST

Building BSTs that are “better-than-balanced.”

Dynamic Graphs

Analyzing graphs as they change in real-time.

Modern Hashing

Breakthroughs by a CS166 alum!

Stringy Thingies

Algorithms for text search and genomics.

Something Else?

What do you want to learn?

Next Time

- ***Disjoint-Set Forests***
  - Tracking what's connected in a graph.
- ***Compression Functions***
  - Iteratively making things smaller.
- ***Analyzing Disjoint-Set Forests***
  - With a beautiful final result!